The Pion in AdS/QCD

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Abstract

Quantum Chromodynamics (QCD), the theory of the strong force describing quark interactions, is notoriously difficult to make calculations with because of properties such as confinement and asymptotic freedom. A model was developed several years ago using the Anti-de-Sitter/Conformal-Field-Theory (AdS/CFT) Correspondence conjectured by Juan Maldacena [6] in terms of String Theory. In this paper, we first describe QCD and its associated chiral symmetry, and then develop Holographic QCD (or AdS/QCD), the model based on the AdS/CFT correspondence. We then describe the phenomenon of pion condensation to exhibit one discrepancy AdS/QCD shows from expectations. Using the model, we then derive the Gell-Mann-Oakes-Renner (GOR) relation for a real chiral condensate:

$$2m_q\sigma = m_\pi^2 f_\pi^2 \tag{1}$$

where m_q is the quark mass, σ a parameter of the model with the quantum numbers of the chiral condensate, m_π the mass of a pion, and f_π the pion decay constant. In the derivation, we make a small detour to define the pion decay constant using the AdS/CFT correspondence. We then relax the assumption that the chiral condensate is real and derive a generalized GOR relation for a complex chiral condensate. After finding some unexpected results in the complex GOR relation, we attempt to find the cause of our problems and try a new field decomposition to reproduce our desired results.

1 Introduction

1.1 Quantum Chromodynamics

Quantum Chromodynamics is the theory of the strong nuclear force which governs quarks and gluons. Quarks are the basic building blocks of hadrons, of which there are two categories: mesons, which have a quark and

an antiquark, for which pions are an example, and baryons, which have three quarks, for which protons and neutrons are examples. Gluons are the force carriers, or gauge bosons, of the strong force and thus govern the interactions between quarks.

The potential for the strong force defines how quarks will interact at different distances and is quite different from better known forces such as electromagnetism; it is these differences which give the strong force its special properties. At small distances between quarks (henceforth r), the potential is relatively Coulombic (like electromagnetism), and is of the approximate form:

$$V(r) = \frac{g^2(r)}{r} \tag{2}$$

Here, g(r) is the QCD coupling strength, which is found through field theory calculations as follows:

$$g^2(k) = \frac{g^2}{1 + \frac{g^2}{(4\pi)^2}(\frac{11}{3}N_c - \frac{2}{3}N_f)ln(\frac{k^2}{M^2})} \tag{3}$$

In these equations, k stands for the momentum, which can taken to be approximately $1/r$ according to the Uncertainty Principle; N_c is the number of colors, which we take to be 3, and N_f is the number of flavors, which we take to be 6 for very high energies (at lower energies, the number of flavors will include only quarks whose masses are below the energy scale considered); g and M are constant parameters of the theory which are fixed by normalization.

One can see that the coupling strength decreases as two quarks come closer together, because here the momentum k grows inversely to the separation r and causes the denominator to become large, bringing the coupling to zero. There is still the $1/r$ behavior in the potential which causes there to be a nonzero force between the quarks, but the coupling strength goes toward zero as the separation goes to zero. This behavior is called asymptotic freedom and is our first special property of QCD. It is much different from what we are used to with electromagnetism, because as two electric charges come closer together there is less charge screening by quantum fluctuations of the vacuum and the coupling strength actually grows.

At large distances, the potential increases linearly with distance because the coupling increases; this is the statement of confinement, another special property of QCD. It is because of this behavior that one can never find a free quark. As a quark and an antiquark become separated by a larger distance, the potential energy between them increases linearly. Ultimately, they reach a point where it is more energetically favorable for the

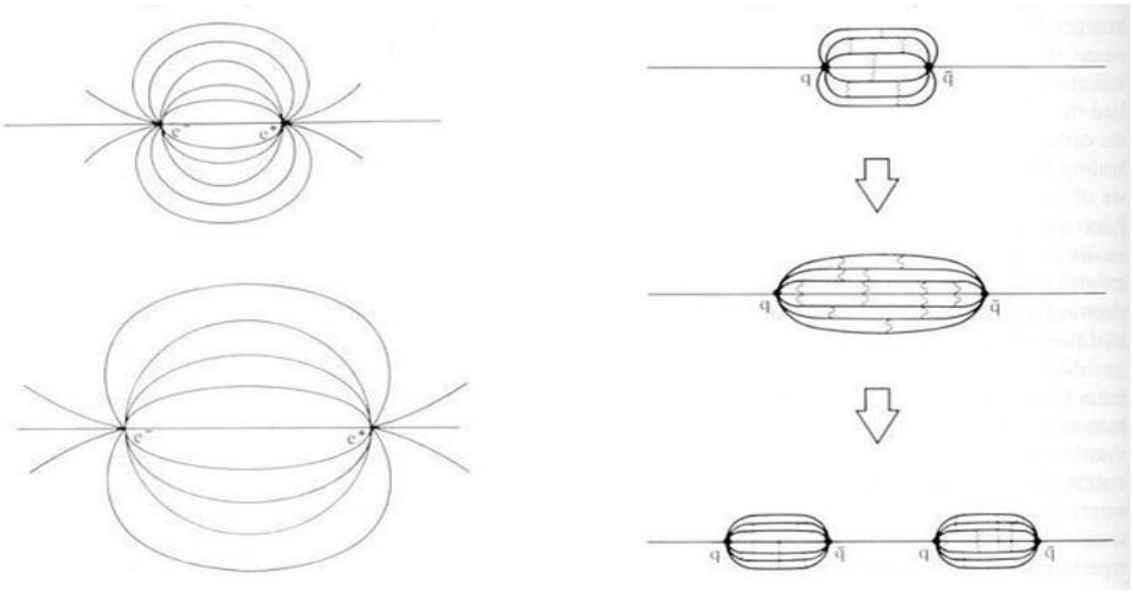

Figure 1: Electric versus Gluomic Fields

Universe to create a new quark-antiquark pair; one binds to each of the original quarks, creating two bound states like we had at the beginning. Thus, a quark can never be found flying freely by itself. Confinement, like asymptotic freedom, is very different from our usual experience with gravity and electromagnetism, in which the force between two objects vanishes as they move further apart. A simple graphical representation of long-range interactions is shown in Figure 1 using field lines to compare electromagnetism's behavior and QCD's behavior; the density of field lines in a particular area shows the energy density in the area.

We now move on to describing QCD in a more technical manner. The Lagrangian density for QCD is as follows:

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \sum_q \bar{q}(i\not{\partial} - g\not{A} - m_q)q \tag{4}$$

where the sum is over all six quarks (u, d, c, s, t, b), g is the coupling strength, and m_q is the quark mass. The term with the sum is a potential term describing the interactions of the quarks. μ and ν are Lorentz indices which run through all 4 spacetime dimensions; we will use the Einstein summation convention in which any repeated index is implicitly summed over all possible values. a here is a gauge index which runs from 1 to the dimensionality of the group- the dimensionality is $n^2 - 1$ for the general SU(n). QCD is SU(3), so we

will have a dimensionality of 8. $F^a_{\mu\nu}$ is called the field strength tensor and is defined as follows:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i f^{abc} A^b_\mu A^c_\nu \tag{5}$$

Here, A_μ is the gauge field whose fluctuations produce the gluons. Thus, we can see that the first term in the Lagrangian is a kinetic term describing the motion of the gluons. f^{abc} are structure constants which are defined by the generators of the group as follows:

$$[T^a, T^b] = i f^{abc} T^c \tag{6}$$

Here, the T matrices are the generators of the group and brackets indicate taking the commutator. The field q in the Lagrangian is a 4-component spinor describing the quark, and $\bar{q}$ is defined in terms of it as $\bar{q} = q^\dagger \gamma^0$. γ^0 here is one of the Dirac gamma matrices, which may be defined as follows:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{7}$$

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} \tag{8}$$

$$\gamma^5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9}$$

The σ matrices are related to the Pauli sigma matrices as $\sigma^\mu = (1, \sigma)$ and $\bar{\sigma}^\mu = (1, -\sigma)$. It is also straightforward to check that $(\gamma^5)^2 = 1$. We can now define the slash notation used in the Lagrangian as $\not{\partial} = \gamma^\mu \partial_\mu$ and $\not{A} = \gamma^\mu A_\mu$.

We can now move on to describe chiral symmetry and show how it arises from the Lagrangian. We will decompose the 4-component spinor q in terms of right-handed and left-handed components as follows:

$$q_L = \left(\frac{1 - \gamma^5}{2} \right) q \tag{10}$$

$$q_R = \left(\frac{1+\gamma^5}{2}\right) q \tag{11}$$

These right- and left-handed components are termed the chirality of the spinor. They correspond well at high energies with the helicity, the direction that the spin of a particle orients itself about its direction of velocity (right-handed is spin parallel to velocity and left-handed is spin anti-parallel). At lower energies, chirality is more abstract and ceases having a physical interpretation such as that, although fields can still be decomposed in terms of chiral components. We can now decompose our Lagrangian in terms of the right- and left-handed spinors as shown:

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \sum_q (\bar{q}_L + \bar{q}_R)(i\partial\!\!\!/ - gA\!\!\!/ - m_q)(q_L + q_R) \tag{12}$$

Let us consider the terms with the slashed partial first. We will need the fact, which can be checked, that $\{\gamma^5, \gamma^\mu\} = 0$, meaning that moving a γ^5 past a γ^μ commutatively picks up a minus sign for that term.

$$i\bar{q_L}\partial\!\!\!/ q_R = i\left(\left(\frac{1-\gamma^5}{2}\right) q\right)^\dagger \gamma^0\gamma^\mu\partial_\mu \left(\frac{1+\gamma^5}{2}\right) q \tag{13}$$

$$= iq^\dagger\gamma^0\gamma^\mu\partial_\mu \left(\frac{1-\gamma^5}{2}\right)\left(\frac{1+\gamma^5}{2}\right) q \tag{14}$$

$$= iq^\dagger\gamma^0\gamma^\mu\partial_\mu \left(\frac{1-(\gamma^5)^2}{4}\right) q \tag{15}$$

$$= 0 \tag{16}$$

The same thing occurs for $i\bar{q_R}\partial\!\!\!/ q_L$. We now consider like terms:

$$i\bar{q_L}\partial\!\!\!/ q_L = i\left(\left(\frac{1-\gamma^5}{2}\right) q\right)^\dagger \gamma^0\gamma^\mu\partial_\mu \left(\frac{1-\gamma^5}{2}\right) q \tag{17}$$

$$= iq^\dagger\gamma^0\gamma^\mu\partial_\mu \left(\frac{1-\gamma^5}{2}\right)\left(\frac{1-\gamma^5}{2}\right) q \tag{18}$$

$$= iq^\dagger\gamma^0\gamma^\mu\partial_\mu \left(\frac{(1-\gamma^5)^2}{4}\right) q \tag{19}$$

$$\neq 0 \tag{20}$$

Thus we see that for the slashed terms ($\not{A}$ will work the same as $\not{\partial}$ because of the similar definition with γ^μ), the same chirality terms survive but the opposite ones vanish. Because the m_q term does not carry a γ^μ with it, the exact opposite occurs: the unlike chiral terms survive but the like ones vanish. Using these chirality arguments, we can simplify our Lagrangian as:

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \sum_q [\bar{q}_L(i\not{\partial} - g\not{A})q_L + \bar{q}_R(i\not{\partial} - g\not{A})q_R - m_q(\bar{q}_L q_R + \bar{q}_R q_L)] \tag{21}$$

Using this form of the Lagrangian we can now show chiral symmetry and its breaking, which will be a vital facet of our model. First, consider only one quark flavor in the Lagrangian above. Consider the following transformations:

$$q_L \rightarrow e^{i\theta_L} q_L \tag{22}$$

$$\bar{q}_L \rightarrow e^{-i\theta_L} \bar{q}_L \tag{23}$$

$$q_R \rightarrow e^{i\theta_L} q_R \tag{24}$$

$$\bar{q}_R \rightarrow e^{-i\theta_L} \bar{q}_R \tag{25}$$

Clear cancellations of the exponentials then occur for the slashed terms. These cancellations do not occur with the m_q term, though, so it is said to explicitly break the chiral symmetry. We can generalize this to all six flavors by using the more general transformation with unitary matrices U as shown below:

$$q_L \rightarrow U_L q_L \tag{26}$$

$$\bar{q}_L \rightarrow U_L^\dagger \bar{q}_L \tag{27}$$

$$q_R \rightarrow U_R q_R \tag{28}$$

$$\bar{q}_R \rightarrow U_R^\dagger \bar{q}_R \tag{29}$$

Similarly, the unitary matrices cancel for the slashed terms but do not for the m_q term. We can now see what is meant by chiral symmetry breaking. These chiral transformations would leave the Lagrangian

unchanged if the quark mass were zero. A nonzero quark mass, though, stops chiral symmetry from being a perfect symmetry for QCD.

There is a theorem in Quantum Field Theory that states that whenever a global symmetry is spontaneously broken, a massless particle is created; these particles are called Goldstone bosons. The operator $\bar{q}q$ has a nonzero vacuum expectation value called the chiral condensate. This chiral condensate is said to spontaneously break chiral symmetry. Chiral symmetry, however, is not a perfect symmetry of QCD, so when it is spontaneously broken, a massive particle, the pion, is created. We will study the pion within our model to delve into the chiral symmetry breaking.

We now make one simplification in our story. The potential for QCD described above suggests that there will be some point we can define where the coupling will become strong and we can no longer consider the quarks free; this critical point is traditionally called Λ_{QCD} and is generally given the value $250MeV$. We can compare our quark masses to Λ_{QCD} to determine which are the most relevant for our study. The lighter quarks will allow us to approximate $m_q = 0$. We will consider only the up and down quarks, which have masses of about $2MeV$ and about $5MeV$ respectively. Considering these two quarks allows us to use the $SU(2) \times SU(2)$ group in our model.

1.2 Yukawa Coupling Terms

Now that we understand chiral symmetry breaking and the basics of QCD, let us move on to describing how the Higgs field, which gives particles mass, is mixed in with our story of QCD, which will be important later. The Standard Model Lagrangian includes terms referred to as Yukawa Coupling terms which describe the interactions of the Higgs field and fermions. These terms are as follows:

$$\mathcal{L}_{Yukawa} = Tr[\sum_{f} \lambda_{f,d} \bar{f_L} H f_{d,R} + \sum_{f} \lambda_{f,u} \bar{f_L} i\sigma^{(2)} H^{\dagger} f_{u,R} + \text{hermitian conjugate}] \tag{30}$$

For us, $\lambda_{f,d} = \lambda_{f,u} = \lambda$, the coupling coefficients, are equal to preserve isospin. H is the Higgs field, which we will decompose as follows, including its vacuum expectation value, v:

$$\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{31}$$

The multiplication by $i\sigma^{(2)}$ (the second Pauli-sigma matrix) in the second term of equation (30) simply flips the two entries so that the vanishing term is the lower one. These expectation values will be multiplied by a unitary matrix U to describe fluctuations of the Higgs field H. We will contain the $\bar{q}_L q_R$ operator as part of a matrix of such operators called Σ. Using all of these prescriptions, we simplify the Yukawa coupling terms as:

$$\mathcal{L}_{Yukawa} = Tr[\lambda \bar{q_i} U_{ij} \langle H \rangle_j d_R + \lambda \bar{q_i} i\sigma^{(2)}_{ik} U^*_{kj} \langle H \rangle_j u_R + \text{hermitian conjugate}] \tag{32}$$

Here, we have identified the fermions we are interested in, the up (u) and down (d) quarks. Multiplying out by the second Pauli sigma matrix on the right term yields:

$$\mathcal{L}_{Yukawa} = Tr[\lambda \bar{q_i} U_{ij} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}_j d_R + \lambda \bar{q_i} U_{ik} \begin{pmatrix} \frac{v}{\sqrt{2}} \\ 0 \end{pmatrix}_k u_R + \text{hermitian conjugate}] \tag{33}$$

$$\mathcal{L}_{Yukawa} = Tr[\lambda \bar{q_i} U_{i2} \left(\frac{v}{\sqrt{2}}\right) q_{R2} + \lambda \bar{q_i} U_{i1} \left(\frac{v}{\sqrt{2}}\right) q_{R1} + \text{hermitian conjugate}] \tag{34}$$

Reconverting these into matrix and column form then gives us:

$$\mathcal{L} = Tr[\frac{\lambda v}{\sqrt{2}} \bar{q_L} U q_R] \tag{35}$$

$$\mathcal{L} = \lambda v Tr[q_R \bar{q_L} U + \text{hermitian conjugate}] \tag{36}$$

Now recall that the operators $\bar{q_L} q_R$ will be contained in a matrix Σ and we wrote our Higgs field out with v as its vacuum expectation value for a U decomposition, allowing us to say $U = H/v$. Combining these, we find the form of the Yukawa coupling terms we will need for later:

$$\mathcal{L} = \lambda Tr[\Sigma^\dagger H + \Sigma H^\dagger] \tag{37}$$

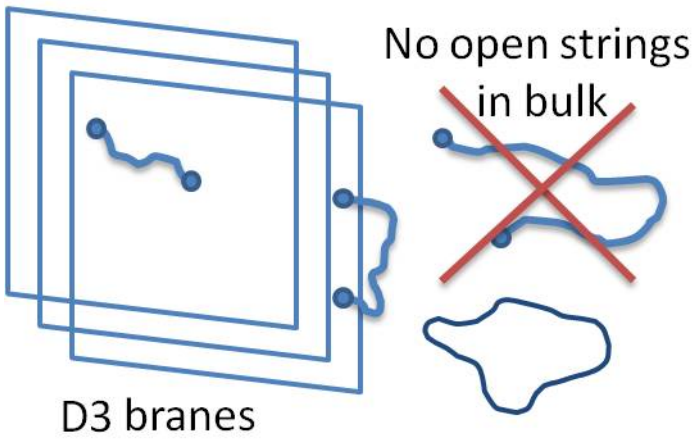

Figure 2: Strings and D3 Branes

1.3 The AdS/CFT Correspondence

We now describe the AdS/CFT correspondence, which will give us a map by which we can create our 5D model of QCD. The AdS/CFT correspondence is borrowed from String Theory, so we will first describe its basic properties using Type IIB String Theory.

String Theory is based around massless, relativistic strings. These strings are incredibly small, so that when one looks at them from far away, one sees that they resolve into a particle. The different modes of a string, which are similar to the modes of a classical string, or the frequencies of an EM wave in a cavity in that they are quantized, correspond to different energies within the string and thus different masses of the resolved particle. Thus, the dynamics of a string, which are based on its action, are also largely dependent on its boundary conditions. Type IIB String Theory contains objects called D3 branes which allow for strings to 'end' on them. The 'D' stands for Dirichlet boundary conditions, so that the position of the endpoints of open strings are fixed on the branes. The '3' signifies that 3 spatial conditions exist for the boundary conditions of a single string on a D3 brane. All open strings must end on D3 branes- there are no strings floating around in the bulk with open-ended boundary conditions. There can be closed strings in the bulk, which would give rise to periodic boundary conditions for the strings. The graviton is contained in the spectrum of these closed string states.

Figure 2 gives an admittedly crude, yet intuitively helpful, grasp of these different conditions. If you

have several D3 branes which are close together, then you can have one or several strings stretched between them as shown in the image. The quantum mechanics of these strings gives rise to what is called $\mathcal{N} = \triangle$ supersymmetric SU(N) Yang-Mills theory in 3+1 dimensions. The N in SU(N) refers to the number of D3 branes involved. This is a special field theory because it is conformal, meaning there is no characteristic scale. As a comparison, QCD is nonconformal, because there is a scale- the Λ_{QCD} parameter that was mentioned earlier gives a characteristic scale to which masses and energies can be compared. Conformal theories are rare (although there are technically an infinite number of them).

Juan Maldacena in [6] considered many D3 branes on top of each other, creating a large source of tension for the strings and thus a large source of energy for them. By doing this, he could consider only classical gravity and ignore quantum gravity fluctuations, simplifying his calculations. Another important consequence of this decision was that the many-brane source now created what are referred to as Ramond-Ramond fields (similar to electromagnetic fields in higher dimensions). These fields then effectively coupled EM and gravity, and when he solved the equations of motion for these coupled theories, he found that his solutions approached $AdS_5 \times S_5$, which describes the large N geometry of the closed bulk strings. The S_5 are small, compact spheres, which teach us about the spectrum of masses that arise, also known as the Kaluza-Klein modes. The AdS_5 isometry group matches to the conformal group of the 3+1 dimensions of the SU(N) theory. Maldacena had effectively linked a 5-dimensional theory with gravity (the bulk) to a 4-dimensional theory without gravity (around the branes).

Maldacena then made a conjecture: A gravitational theory in $d + 1$ dimensions can be equivalent to a theory without gravity in d dimensions. Following his conjecture, other theorists then began to try to create a 'dictionary' which would relate aspects of the d-dimensional theory to the $(d + 1)$-dimensional theory. Witten, and separately Gubzer, Klebanov, and Polyakov figured out this dictionary, which I will explain in part below. The AdS/CFT dictionary will be our main tool, allowing us to take QCD in 4D and build a model in 5D, which will have limitations as I will describe, but will still be reasonably accurate. Hence, the AdS/CFT correspondence is an incredibly powerful tool in model-building. The dictionary, whittled down to what we will need for our specific model, is shown in Table 1.

Properties/Objects in 3+1 dimensions	Properties/Objects in 4+1 AdS_5 dimensions
Conformal Symmetry	Isometries of AdS_5
Symmetries	Gauge Fields
Confinement	Spacetime becomes compact (Slice of AdS)
Operator	Field
Dimension of Operator	Mass of Field
Source for Operator	Non- Normalizable Mode of Field
Vacuum Expectation Value of Operator	Background of Normalizable Mode

Table 1: AdS/CFT Correspondence Dictionary

2 Holographic QCD

We now have all the tools we need to build the model, called Holographic QCD or AdS/QCD. We will use the dictionary of the AdS/CFT correspondence despite the fact that QCD is not conformal. We still use AdS, though, because it is a simple geometry to work with. Also, QCD seems to become conformal at high energies, which may give us some added accuracy.

From the dictionary entry about confinement, we take the spacetime to be a slice of AdS_5. Symmetries which exist within the 4D model will give us our gauge fields. We found that a symmetry occurred for the term $\bar{q}_L \gamma^\mu T^a q_L$, so we obtain a gauge field equivalent for this term which we will call $L^{\mu a}$; similarly, we have a $R^{\mu a}$. We want to describe the physics brought about by chiral symmetry and its breaking, which occurs (as demonstrated above) because of the operator $\bar{q}_L q_R$. Thus, we promote the $\bar{q}_L q_R$ operator to a field, which we will call X.

We can obtain a mass for the field X by finding the dimension of the operator $\bar{q}_L q_R$. We know that the action must be dimensionless; the action is simply $S = \int d^4x \mathcal{L}$. Thus, we must be able to cancel four length dimensions using our Lagrangian. The Lagrangian for QCD contains the terms $\bar{q}(i\partial\!\!\!/ - m)q$. Thus, we have a dimensionality of $2[q] + 1 = 4$ where $[q]$ is the dimension of the q operator. We then see $[q] = 3/2$, which allows us to find that $[\bar{q}q] = 3$. The mass of the field is then found from the dimension of $\bar{q}_L q_R$ according to $R^2_{AdS} M^2_X = [\bar{q}q]([\bar{q}q] - 4)$. We then have that the mass of the field X is:

$$m_X^2 = \frac{-3}{R_{AdS}^2} \tag{38}$$

Here, m_X is the mass of the field X and R_{AdS} is the curvature of AdS_5. It may seem strange that a mass squared is negative, but it has been shown that the requirement for stability is $m_X^2 > \frac{-4}{R_{AdS_5}}$, which is called the Breitenlohner-Freedman bound [5]. We will usually set $R_{AdS} = 1$.

We can now solve the equations of motion for the field X with all of the gauge fields set to 0 to obtain the background for X:

$$X_0(z) = \frac{1}{2} m_q z + \frac{1}{2} \sigma z^3 \tag{39}$$

Fluctuations about this background describe the Kaluza-Klein modes of the system which in turn describe the bound states of QCD.

We now have all of the information we need to write the action for our model. I will describe each term and where it arises below, but here it is in full:

$$S = \int d^5x \sqrt{g} \left(-\frac{1}{4g_5^2} L^a_{MN} L^{MNa} - \frac{1}{4g_5^2} R^a_{MN} R^{MNa} + Tr[D_M X^\dagger D^M X] + 3Tr[|X|^2] \right) \tag{40}$$

This is our action for Holographic QCD. Capital letters represent sums over all 5 dimensions of AdS_5. We will be calling the fifth dimension z, and sums over only the 4 'regular' dimensions will be denoted by lowercase Greek letters (μ, ν). The integral is over the five spatial dimensions of AdS_5, giving the d^5x. We must now guarantee that d^5x is invariant under coordinate transformations. The $\sqrt{g}$ does just this; it is the square root of the determinant of the metric. The metric is given by:

$$g_{MN} = \frac{1}{z^2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{41}$$

We can use g_{MN} to demonstrate manipulations of the first two terms in parentheses in the action,

equation (40). We have $L^a_{MN}L^{MNa} = L^a_{MN}L^a_{AB}g^{MA}g^{NB}$ and $L^a_{MN} = T^aL^a_{MN}$, where T^a is the generator of the group for the given a gauge index. We will often write these terms as $L^a_{MN}L^{MNa} = F_L^2$, because they take the place of the field strength tensor in the original QCD Lagrangian and are essentially kinetic terms for the gauge fields. They are fixed by our $SU(2) \times SU(2)$ gauge invariance.

We now come to the kinetic term for the X field. We define the covariant derivative here as:

$$D_M X = \partial_M X - iL_M X + iXR_M \tag{42}$$

The $3Tr[|X|]^2$ is the mass term for the field, whose coefficient is fixed by the $m_X^2 = -3$ which we found earlier. We will use these definitions with equation (40) to derive the GOR relation.

3 Pion Condensation

We will now take a slight detour and explain pion condensation, an aspect of Holographic QCD that differs from our expectations and provides some motivation for looking into the subtleties of the model to find where some modifications may need to be made. The Chiral Lagrangian is another model of QCD which is also based on chiral symmetry breaking, allowing us to compare it to our results. The Chiral Lagrangian contains the terms:

$$\mathcal{L} = \frac{1}{4}f_\pi^2 Tr[(D_\mu U)(D^\mu U^\dagger)] - \frac{m_\pi^2 f_\pi^2}{4}Tr[U + U^\dagger] \tag{43}$$

These terms contain phenomenological data, the mass of the pion and the pion decay constant. The matrix U contains information about the pion field, $U = exp(2i\pi^a T^a)$. We wish to consider systems with large isospin (many more up quarks than down quarks or vice versa) and very low temperatures (since our model assumes $T = 0$). A physical example of this environment would be in a neutron star, where the temperatures are very low and there are essentially only neutrons, which are udd, meaning the system has many more down quarks than up quarks, adding to create a large isopsin. To govern the amount of isospin in the system, we introduce an isospin chemical potential μ_I. From statistical mechanics, we know that the potential in our Lagrangian should then have a term $V_{eff} = \mu_I N_{I3}$. N_{I3} is the number density for eigenvalues of the third generator. For definiteness, we have chosen the third generator of the group; this is

completely general since we can always perform a rotation as necessary. It is a convenient choice because the third generator is diagonal, with eigenvalues for the pions according to:

$$T_3 \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} = \begin{pmatrix} \pi^+ \\ 0 \\ -\pi^- \end{pmatrix} \tag{44}$$

We can relate the isospin number density to the quark field as:

$$N_{I3} = q^\dagger T_3 q \tag{45}$$

We can perform a slight reidentification of N_{I3} by multiplying by $\gamma^0\gamma^0$, which is equal to the identity matrix. We then have $q^\dagger T_3 q = q^\dagger \gamma^0\gamma^0 T_3 q = \bar{q}\gamma^0 T_3 q$. The conserved current corresponding to the isospin symmetry is given by $J^{\mu 3} = \bar{q}\gamma^\mu T_3 q$. Thus, we see that the zeroth component of the conserved current is in fact itself the isospin number density. The Lagrangian already contains terms such as $V_\mu J^{\mu 3}$, so to add our $\mu_I N_{I3}$ term, we simply need to turn on a background for the time component of the potential V_μ. We will turn on a background by identifying the covariant derivative as:

$$D_0 U = \partial_0 U - \frac{\mu_I}{2}(T_3 U - U T_3) \tag{46}$$

$$D_i U = \partial_i U \tag{47}$$

Using these new covariant identifications, we find that the potential for U is:

$$V_{eff} = \frac{f_\pi^2 \mu_I^2}{8} Tr(T_3 U T_3 U^\dagger - 1) - \frac{f_\pi^2 m_\pi^2}{2} Tr(U + U^\dagger) \tag{48}$$

An effective ansatz for the field U that is followed in [3], which I will use as well, is

$$U = cos\alpha + i(T_1 cos\phi + T_2 sin\phi) sin\alpha \tag{49}$$

Here, α and ϕ are simply parameters of the ansatz and T_1 and T_2 are the first and second generators. If

we substitute the U ansatz into the effective potential equation above, we find:

$$V_{eff} = \frac{f_\pi^2 \mu_I^2}{4}(cos(2\alpha) - 1) - f_\pi^2 m_\pi^2 cos\alpha \tag{50}$$

As one can see, the ϕ dependence cancels out in the resulting potential. Minimizing the potential with respect to α reveals two regions. When $|\mu_I| < m_\pi$, we find $\alpha = 0$ so that the resulting pion matrix $U = 1$. The isospin density here is zero because it costs $m_\pi - |\mu_I|$ to excite the lowest-energy pion, so no pions are excited. The other case occurs when $|\mu_I| > m_\pi$, in which case the minimum occurs at

$$cos\alpha = \frac{m_\pi^2}{\mu_I^2} \tag{51}$$

Here, pions are excited, creating a Bose condensate. Interactions between pions stabilize the system, allowing us to find the equilibrium number density by differentiating our Lagrangian with respect to μ_I. We thus find:

$$N_{I3} = -\frac{\partial \mathcal{L}}{\partial \mu_I} = f_\pi^2 \mu_I sin^2\alpha = f_\pi^2 \mu_I \left(1 - \frac{m_\pi^4}{\mu_I^4}\right) \tag{52}$$

This provides a quantitative prediction for the density of pions in a pion condensate based on phenomenological factors like the pion decay constant and mass. There is an analogous approach using 5D Holographic QCD which is beyond the scope of my thesis, but it does produce some suspicious results which I will now discuss. First, one can show that in the 4D analysis the phase transition for a pion condensate is a second order transition (*i.e.* derivatives of the order parameter are discontinuous across the transition, but the order parameter itself is continuous). The 5D model, however, finds a first order transition, which is qualitatively different from what is expected based on the 4D theory. This is the first indication of what we shall find later, that Holographic QCD is at times inconsistent with expectations, although it does on the whole produce good results.

4 Approximating the Action

The action for the model as shown above is:

$$S = \int d^5x\sqrt{g}Tr\{|DX|^2 + 3\,|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2)\} \tag{53}$$

In review, definitions for these functions within the model are as follows:

$$D_M X = \partial_M X - iL_M X + iXR_M \tag{54}$$

$$F_{MN} = \partial_M A_N - \partial_N A_M + f^{abc}A_M^b A_N^c \tag{55}$$

We use the convention that the four original spacetime coordinates are signified by lower-case Greek indices (*i.e.* μ, ν), the fifth-dimension parameter is z, and all five dimensions summed over are represented by capital letters (*i.e.* M, N). Since we are only considering the linearized equations of motion, the term with the structure constants will be thrown out. We may now expand the action in terms of these functions to find:

$$\begin{aligned} S = \int d^5x\sqrt{g}Tr\{|\partial_M X|^2 - iL_M^b(\partial^M X^\dagger)T^b X + iR_M^b(\partial^M X^\dagger)XT^b + L_M^a L^{Mb}X^\dagger T^a T^b X + \\ iL_M^a X^\dagger T^a(\partial^M X) - L_M^a R^{Mb}X^\dagger T^a X T^b - iR_M^a T^a X^\dagger(\partial^M X) - \\ R_M^a L^{Mb}T^a X^\dagger T^b X + R_M^a R^{Mb}T^a X^\dagger X T^b + 3\,|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2)\} \end{aligned} \tag{56}$$

Let us first consider the term $|\partial_M X|^2 = (\partial_M X)(\partial^M X)$. The field X is given by $X = X_0 exp(2i\pi^c T^c)$. Note that the exponential matrix is unitary, so we can also write $X = X_0 U$, where U represents the unitary matrix and $X_0 = \frac{1}{2}v(z)$ is a background which only depends on the fifth dimension coordinate z. We can evaluate $Tr\{|\partial_M X|^2\}$ in terms of the unitary matrix and the background to find:

$$Tr\{|\partial_M X|^2\} = \frac{1}{4}|(\partial_M v)U + v(\partial_M U)|^2 \tag{57}$$

$$Tr\{|\partial_M X|^2\} = \frac{1}{4}(|\partial_M v|^2 U^\dagger U + v(\partial_M v^*)U^\dagger(\partial_M U) + v^*(\partial_M v)(\partial_M U^\dagger)U + |v|^2|\partial_M U|^2) \tag{58}$$

We note that because the matrix U is unitary, we have that the first term is proportional to the identity and serves as the background, so we neglect it. Secondly, since we have $v(z) \in \Re$, the second two terms cancel. We thus have:

$$Tr\{|\partial_M X|^2\} = \frac{1}{4} Tr\{|v|^2 |\partial_M U|^2\} \tag{59}$$

We can now evaluate by expanding according to:

$$U \approx 1 + 2i\pi^c T^c - 2\pi^c T^c \pi^b T^b \tag{60}$$

The background field term $v(z)$ is simply a function of our fifth-dimension parameter, z:

$$v(z) = m_q z + \sigma z^3 \tag{61}$$

We now contract the pion indices with the metric to add a factor of z^2 and raise all indices to the top. We also keep only terms in the expansion to quadratic order in the pion fields, which yields:

$$Tr\{|\partial_M X|^2\} = \frac{1}{4}(m_q^2 z^2 + 2m_q \sigma z^4 + \sigma^2 z^6) Tr\{4(\partial_M \pi^c) T^c (\partial_M \pi^b) T^b\} \tag{62}$$

Now, we note that $Tr\{T^a T^b\} = \frac{1}{2}\delta^{ab}$, which allows us to find a final expression for $Tr\{|\partial_M X|^2\}$ (converting all dummy indices into a after combining them):

$$Tr\{|\partial_M X|^2\} \approx \frac{1}{2} m_q^2 z^4 (\partial_M \pi^a)(\partial_M \pi^a) + m_q \sigma z^6 (\partial_M \pi^a)(\partial_M \pi^a) + \frac{1}{2}\sigma^2 z^8 (\partial_M \pi^a)(\partial_M \pi^a) \tag{63}$$

We now examine the terms involving L and R:

$$\begin{aligned} S \supset Tr\{ & -iL_M^b (\partial_M X^\dagger) T^b X + iR_M^b (\partial_M X^\dagger) X T^b + L_M^a L_M^b X^\dagger T^a T^b X \\ & + iL_M^a X^\dagger T^a (\partial_M X) - L_M^a R_M^b X^\dagger T^a X T^b - iR_M^a T^a X^\dagger (\partial_M X) \\ & - R_M^a L_M^b T^a X^\dagger T^b X + R_M^a R_M^b T^a X^\dagger X T^b)\} \end{aligned} \tag{64}$$

Let us first look at the terms without derivatives of X, which we will denote by $\Gamma_{No\ \partial}$, again contracting to make all indices upper:

$$\begin{aligned}\Gamma_{No\ \partial} &= Tr\{L_M^a L_M^b X^\dagger T^a T^b X + -L_M^a R_M^b X^\dagger T^a X T^b - R_M^a L_M^b T^a X^\dagger T^b X + R_M^a R_M^b T^a X^\dagger X T^b\} \\ &= \frac{z^2}{2}[L_M^a L_M^a X^\dagger X + -L_M^a R_M^a X^\dagger X - R_M^a L_M^a X^\dagger X + R_M^a R_M^a X^\dagger X] \\ &= \frac{z^2}{2}\left|X\right|^2 (L_M^a - R_M^a)^2 \end{aligned} \tag{65}$$

The definitions of the axial and vector portions of the fields are as follows:

$$A_M^a = \frac{L_M^a - R_M^a}{2} \tag{66}$$

$$V_M^a = \frac{L_M^a + R_M^a}{2} \tag{67}$$

We thus observe that the terms currently being considered contain only axial parts of the field. Also, $\left|X\right|^2 = \left|X_0\right|^2 \left|U\right|^2 = \left|X_0\right|^2$. We thus write:

$$\Gamma_{No\ \partial} = 2z^2 \left|X_0\right|^2 A_M^a A_M^a \tag{68}$$

We now consider the second set of L, R terms, those containing derivatives of the X field, which we denote Γ_∂:

$$\begin{aligned}\Gamma_\partial &= Tr\{-iL_M^b(\partial_M X^\dagger)T^b X + iR_M^b(\partial_M X^\dagger)XT^b + iL_M^a X^\dagger T^a(\partial_M X) - iR_M^a T^a X^\dagger(\partial_M X)\} \\ &= iz^2 Tr\{((\partial_M X)X^\dagger - (\partial_M X^\dagger)X)(L_M^b - R_M^b)T^b\} \\ &= 2iz^2 Tr\{((\partial_M X)X^\dagger - (\partial_M X^\dagger)X)A^{Mb}T^b\} \end{aligned} \tag{69}$$

We again expand the field X in terms of the exponential and cancel terms in order to find:

$$\Gamma_{\partial} = 2iz^2 Tr\{A^{Ma}T^aT^c(4i\,|X_0|^2)(\partial_M\pi^c)\}$$
$$= -4z^2 A^{Ma}(\partial_M\pi^a)\,|X_0|^2 \tag{70}$$

The L,R terms as a whole then give us:

$$\Gamma_{No\ \partial} + \Gamma_{\partial} =$$
$$\frac{1}{2}z^4 m_q^2 A_M^a A^{Ma} + m_q\sigma z^6 A_M^a A^{Ma} + \frac{1}{2}\sigma^2 z^8 A_M^a A^{Ma}$$
$$-z^4 m_q^2 A^{Ma}(\partial_M\pi^a) - 2z^6 m_q\sigma A^{Ma}(\partial_M\pi^a) - \sigma^2 z^8 A^{Ma}(\partial_M\pi^a) \tag{71}$$

Finally, we consider the last three terms in equation (53). $3\,|X|^2$ is proportional to the identity matrix because $X = X_0U$ is unitary; this term thus only contributes to the background, so we neglect it. We evaluate $F_L^2 + F_R^2$ by first separating the dependence of the regular four spacetime dimensions and the extra fifth dimension coordinate, again contracting indices:

$$\frac{-1}{4g_5^2}Tr\{F_L^2 + F_R^2\} = \frac{-z^4}{4g_5^2}((F_L^{\mu\nu}F_{L\mu\nu} + 2F_L^{\mu z}F_{L\mu z}) + (F_R^{\mu\nu}F_{R\mu\nu} + 2F_R^{\mu z}F_{R\mu z})) \tag{72}$$

We now evaluate to quadratic order in the pion fields, making use of our gauge choice that $A_z = 0$ to simplify the z portion of the expression:

$$\frac{-1}{4g_5^2}Tr\{F_L^2 + F_R^2\} = \frac{-z^4}{4g_5^2}[(\partial_\mu L_\nu - \partial_\nu L_\mu)(\partial^\mu L^\nu - \partial^\nu L^\mu) - 2(\partial_z L_\mu)(\partial^\mu L^z) +$$
$$(\partial_\mu L_\nu - \partial_\nu L_\mu)(\partial^\mu L^\nu - \partial^\nu L^\mu) - 2(\partial_z L_\mu)(\partial^\mu L^z)] \tag{73}$$

We now recall the equations relating A and V with L and R, equations (66) and (67) above. We now multiply everything out and substitute into our expression to acquire a new one in terms of the axial and vector parts of the field, A and V:

$$\frac{-1}{4g_5^2}Tr\{F_L^2+F_R^2\}=\frac{-z^4}{4g_5^2}[(\partial_\mu V_\nu+\partial_\mu A_\nu)(\partial^\mu V^\nu+\partial^\mu A^\nu)+(\partial_\nu V_\mu+\partial_\nu A_\mu)(\partial^\nu V^\mu+\partial^\nu A^\mu)$$
$$-2(\partial_\mu V_\nu+\partial_\mu A_\nu)(\partial^\nu V^\mu+\partial^\nu A^\mu)-2(\partial_z V_\mu+\partial_z A_\mu)(\partial^z V^\mu+\partial^z A^\mu)+(\partial_\mu V_\nu-\partial_\mu A_\nu)(\partial^\mu V^\nu-\partial^\mu A^\nu)$$
$$+(\partial_\nu V_\mu-\partial_\nu A_\mu)(\partial^\nu V^\mu-\partial^\nu A^\mu)-2(\partial_\mu V_\nu-\partial_\mu A_\nu)(\partial^\nu V^\mu-\partial^\nu A^\mu)-2(\partial_z V_\mu-\partial_z A_\mu)(\partial^z V^\mu-\partial^z A^\mu)] \quad (74)$$

We may now combine terms and simplify to find:

$$\frac{-1}{4g_5^2}Tr\{F_L^2+F_R^2\}=\frac{-z^4}{4g_5^2}[4(\partial_\mu V_\nu)(\partial^\mu V^\nu)+4(\partial_\mu A_\nu)(\partial^\mu A^\nu)-4(\partial_\mu V_\nu)(\partial^\nu V^\mu)$$
$$-4(\partial_\mu A_\nu)(\partial^\nu A^\mu)-4(\partial_z V_\mu)(\partial^z V^\mu)-4(\partial_z A_\mu)(\partial^z A^\mu)] \quad (75)$$

If we consider only the axial terms, which we will do for our derivation of the GOR relation, further simplifications occur:

$$\frac{-1}{4g_5^2}Tr\{F_L^2+F_R^2\}=\frac{-z^4}{4g_5^2}F_A^aF_A^a \quad (76)$$

We can now write the final form of our approximated action by summing all of these results. Note that the factor of $\sqrt{g}$ in equation (53), the original action, is the square root of the determinant of the metric, which means $\sqrt{g}=\frac{1}{z^5}$. The approximated action is:

$$S=\int d^5x[\frac{1}{2z}m_q^2(\partial_M\pi^a)(\partial_M\pi^a)+m_q\sigma z(\partial_M\pi^a)(\partial_M\pi^a)+\frac{1}{2}\sigma^2z^3(\partial_M\pi^a)(\partial_M\pi^a)+\frac{1}{2z}m_q^2A_M^aA^{Ma}$$
$$+m_q\sigma zA_M^aA^{Ma}+1/2\sigma^2z^3A_M^aA^{Ma}-\frac{1}{z}m_q^2A^{Ma}(\partial_M\pi^a)-2zm_q\sigma A^a(\partial_M\pi^a)$$
$$-\sigma^2z^3A^{Ma}(\partial_M\pi^a)-\frac{1}{4g_5^2z}F_A^aF_A^a] \quad (77)$$

Most of these terms can be factored together to give a simpler expression:

$$S=\int d^5x\left[-\frac{1}{4g_5^2z}F_A^aF_A^a+\frac{v(z)^2}{2z^3}(\partial_M\pi^a-A_M^a)^2\right] \quad (78)$$

5 Deriving the Equations of Motion

We have the approximated action:

$$S = \int d^5x \left[\frac{-1}{4g_5^2 z} F^a_{MN} F^{MNa} + \frac{v(z)^2}{2z^3} (\partial_M \pi^a - A^a_M)^2 \right] \tag{79}$$

The Euler-Lagrange equations are given by:

$$\partial_M \left(\frac{\partial L}{\partial(\partial_M A^a_N)} \right) = \frac{\partial L}{\partial A^a_N} \tag{80}$$

Calculating these derivatives for the A field gives:

$$\partial_M \left(\frac{1}{z} F^{MNa} \right) = \frac{g_5^2 v(z)^2}{z^3} (\partial^N \pi^a - A^{Na}) \tag{81}$$

We now separate this equation into its transverse and longitudinal components. First, we consider the transverse. The above equation yields:

$$-\partial_z \left(\frac{1}{z} (\partial_z A^\nu - \partial^\nu A_z) \right) + \frac{1}{z} \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{g_5^2 v(z)^2}{z^3} A^\nu \tag{82}$$

We now impose our gauge choices that $A^z = 0$ and $\partial_\mu A^\mu = 0$, which give us:

$$-\partial_z \left(\frac{1}{z} (\partial_z A^\nu) \right) + \frac{1}{z} \partial_\mu (\partial^\mu A^\nu) = \frac{g_5^2 v(z)^2}{z^3} A^\nu \tag{83}$$

We now make an ansatz that $A^\mu(x, z) = e^{-iq \cdot x} A^\mu(q, z)$. Plugging our ansatz in, taking the derivatives, and canceling the exponentials yields:

$$\partial_z \left(\frac{1}{z} \partial_z A^\mu(q, z) \right) + \frac{q^2}{z} A^\mu(q, z) - \frac{v^2 g_5^2}{z^3} A^\mu(q, z) = 0 \tag{84}$$

This is our equation of motion for the transverse part of the field. We now derive the longitudinal part of the field, assuming $A^{Na} = \partial^N \phi^a$. We then have the linear part of $F^{\mu\nu} = 0$ by the equality of mixed partials according to:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \partial^\mu(\partial^\nu \phi) - \partial^\nu(\partial^\mu \phi) = 0 \tag{85}$$

The equation of motion (81) then becomes:

$$\partial_z \left(\frac{1}{z} \partial^z \partial^\nu \phi^a \right) = \frac{g_5^2 v(z)^2}{z^3} (\partial^N \pi^a - \partial^N \phi^a) \tag{86}$$

We make a similar ansatz as previously, letting $\phi^a(x,z) = \phi^a(q,z)e^{-iq\cdot x}$ and $\pi^a(x,z) = \pi^a(q,z)e^{-iq\cdot x}$. The ansatz yields our equation of motion for the longitudinal field ϕ:

$$\partial_z \left(\frac{1}{z} \partial^z \partial^\nu \phi^a(q,z) \right) + \frac{g_5^2 v(z)^2}{z^3} (\pi^a - \phi^a(q,z)) = 0 \tag{87}$$

We now seek an equation of motion for π. The only term in the action containing π is:

$$S = \int d^5x \frac{v(z)^2}{2z^3} (\partial_M \pi^a - A_M^a)^2 = \int d^5x \frac{v(z)^2}{2z^3} [(\partial_z \pi^a)^2 + (\partial_\mu \pi^a - \partial_\mu \phi^a)^2] \tag{88}$$

The Euler-Lagrange equations, along with our ansatz, give:

$$\partial_z \left(\frac{1}{z} \partial^z \pi^a \right) = \frac{q^2 v(z)^2}{z^3} (\pi^a - \phi^a(q,z)) \tag{89}$$

We can produce a more useful version of this equation if we combine it with equation (87), which yields:

$$\frac{g_5^2 v(z)^2}{z^2} \partial_z \pi^a = q^2 \partial_z \phi^a \tag{90}$$

6 Defining the Pion Decay Constant

We now take a small digression in order to derive an expression for the decay constant found in the GOR Relation. The pion decay constant is measured by observing pion decays which occur via the weak interactions; pions can decay into electrons, muons, positrons (anti-electrons), or anti-muons. The relevant terms in the weak Hamiltonian are:

$$\mathcal{H}_W = \frac{G_f}{\sqrt{2}} V_{ud} \bar{\psi}_d \gamma_\lambda (1+\gamma_5) \psi_u [\bar{\psi}_{\nu_e} \gamma^\lambda (1+\gamma^5) \psi_e + \bar{\psi}_{\nu_\mu} \gamma^\lambda (1+\gamma^5) \psi_\mu] \tag{91}$$

One can see here an expression similar to our conserved current $J = \bar{q}\gamma^\mu\gamma^5 q$ as well as other constants measurable by other means. The $\pi^+ \to e^+\nu_e$ decay has an invariant amplitude which is:

$$\mathcal{M}_{\pi^+ \to e^+\nu_e} = -G_F V_{ud} f_\pi m_\mu \bar{u}_\nu (1-\gamma^5) v_\mu \tag{92}$$

Here we see the pion decay constant has appeared. Our final step then brings us to the decay rate:

$$\Gamma_{\pi^+ \to e^+\nu_e} = \frac{G_F^2}{4\pi} f_\pi^2 m_\mu^2 m_\pi |V_{ud}|^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 \tag{93}$$

Analogous expressions exist for the other particles the pions can decay into. Our first observation is that these decay rates are dependent on the mass of the resultant particle. The mass dependence is part of the reason that pions usually decay into muons: they are simply heavier than electrons, increasing the likelihood of pions decaying into them, but not so heavy that they cost too much energy to create, as tau particles do. The pion decay constant can be measured by studying the decays of pions into these particles and extracting the decay constant from the other constants which can be measured in other ways. Thus, the pion decay constant gives information about the rates and amplitudes of pion decays.

We now wish to derive the form of the pion decay constant in terms of the fields within our 5D model. First, we note that the decay constant is defined by:

$$\langle 0 | J_A^{\mu a}(x) | \pi^b(q) \rangle = f_\pi e^{-iqx} q^\mu \delta^{ab} \tag{94}$$

We will use a slightly different function to evaluate the decay constant, though, the two-point current function:

$$\langle 0 | T[J_A^{\mu a}(x) J_A^{\nu b}(0)] | 0 \rangle \tag{95}$$

Consider inserting a complete set of states between the two currents in the expression above. There are only two classes of single-particle states which would have nonzero expectation values with these axial

currents: pions and a_1 mesons. The pion portion give a contribution to the two point current function of $f_\pi^2 q^\mu q^\nu e^{iqx}\delta^{ab}$. The a_1 mesons are a little different in that they contain a polarization over which we must sum as well. Their single current expectation value evaluates to:

$$\langle 0|J_A^{\mu a}(x)|a_1^{\mu b}(\epsilon,q)\rangle = f_{a_1}\epsilon^\mu e^{iqx}\delta^{ab} \tag{96}$$

Here, ϵ represents the polarization states. We choose a basis of polarization states according to:

$$\sum_{r=1}^{3}\epsilon_r^\mu(q)\epsilon_r^\nu(q)^* = g_{\mu\nu} - \frac{q_\mu q_\nu}{m_n^2} \tag{97}$$

Using this basis, we may find what the a_1 contribution to the two-point function is after we insert the complete set of states:

$$\langle 0|T[J_A^{\mu a}\left(\sum |a_1\rangle\langle a_1|\right)J_A^{\nu b}(0)]|0\rangle = f_{a_1}^2\frac{g_{\mu\nu}q^2 - q_\mu q_\nu}{(q^2-m_n^2)m_n^2} \tag{98}$$

We now define the evaluation to be equal to:

$$\int_x e^{iqx}\langle 0|T[J_A^{\mu a}\left(\sum states\right)J_A^{\nu b}(0)]|0\rangle = \Pi_A(q^2)(g_{\mu\nu}q^2 - q_\mu q_\nu) \tag{99}$$

The action for the axial and vector fields are equal except for one term which comes along for the ride in the axial case; for this reason, now that the basic definitions are established, I will consider the vector field V for simplicity and then switch back to A at the end without any problems. The crux of our problem has now become evaluating the two-point current so that we may pull out the pion decay constant portion. The two-point current function evaluation is defined as:

$$\langle 0|T[J_V^{\mu a}\left(\sum states\right)J_V^{\nu b}(0)]|0\rangle = \frac{1}{i^2 Z}\frac{\delta^2 Z[V_\mu^0(x)]}{\delta V_\mu(x)\delta V_\nu(y)} \tag{100}$$

Here, Z is the generating functional for current correlators. We have defined $V_\mu^0(x)$ as our boundary vector field according to $V_\mu^0(x) = V_\mu(x,z)|_{z=\epsilon}$ where ϵ is the location of one of the walls of our 5th dimension which is intended to eventually be taken to 0. The AdS/CFT correspondence provides our next step. According to AdS/CFT, the generating functional Z is related to the 5D action as follows:

$$Z[V_\mu^0(x)] = e^{iS_{5d}[V_\mu^0(x)]} \tag{101}$$

Taking the variational derivates of Z as shown above then gives us the two point current function in terms of the 5D action:

$$\langle 0|T[J_V^{\mu a}\left(\sum states\right)J_V^{\nu b}(0)]|0\rangle = \frac{\delta^2 S_{5D}[V_\mu^0(x)]}{\delta V_\mu(x)\delta V_\nu(y)} \tag{102}$$

Let us now find a simplified version of the action for the vector field so we may take these derivatives. The only terms in the action which include V are:

$$S_V = \frac{-1}{4g_5^2}\int \frac{d^5x}{z^5} z^4(\partial_M V_N - \partial_N V_M)(\partial_P V_Q - \partial_Q V_P)\eta^{MP}\eta^{NQ} \tag{103}$$

We can separate out the z portion and the other four dimensions and take the equation of motion using the Euler-Lagrange equation to find:

$$\frac{-1}{g_5^2}\left(\frac{1}{z}\partial_\mu(\partial_\mu V^\nu - \partial^\nu V^\mu) + \partial_z\left(\frac{1}{z}(\partial^z V^\nu - \partial^\nu V^z)\right)\right) = 0 \tag{104}$$

We now impose our gauge choice $V^z = 0$ and Fourier Transform so that $V^\mu(x,z) = e^{iqx}V^\mu(q,z)$. We are also considering the transverse case, since this is the only portion that couples to a conserved current, so that $\partial^\nu V^\mu = 0$. The equation of motion then becomes:

$$\frac{-1}{g_5^2}\left(\partial_z\left(\frac{1}{z}\partial^z V^\nu(q,z)\right) - \frac{1}{z}q^2 V(q,z)\right) = 0 \tag{105}$$

We may now notice a convenient simplification which occurs in our action. If we integrate the action by parts, we reproduce the equation of motion exactly, so that we may take the action to zero except for a surface term. The surface term must be made transverse, so we will add a transverse projection operator. This gives our action as:

$$S = \frac{-1}{2g_5^2}\int d^4q\left(\frac{1}{zq^2}V_\mu^0(q)\partial_z V^\mu(q,z)\right)(q_\mu q_\nu - q^2 g_{\mu\nu}) \tag{106}$$

We can now take our derivates as we wish. The only step left is that our action is Fourier transformed

to depend on momentum but the derivates are with respect to position-dependent vector fields. Suppose we can decompose our vector field using a boundary field multiplied by a bulk-to-boundary propagator: $V_\mu(q,z) = V_\mu^0(q)V(q,z)$. We may then Fourier transform our action according to:

$$S = \frac{-1}{2g_5^2}\int \frac{d^4q d^4x_1 d^4x_2}{zq^2} e^{iq(x_1+x_2)}\delta^4(x-x_1)\delta^4(y-x_2)V_\mu^0(x_1)\partial_z[V_0^\mu(x_2)V(q,z)](q_\mu q_\nu - q^2 g_{\mu\nu}) \tag{107}$$

We can then take the derivatives prescribed by equation (102) to find:

$$\frac{\delta^2 S_{5D}[V_\mu^0(x)]}{\delta V_\mu(x)\delta V_\nu(y)} = \frac{-1}{g_5^2}\int \frac{d^4q}{zq^2} e^{iqx}\partial_z V(q,z)(q_\mu q_\nu - q^2 g_{\mu\nu}) \tag{108}$$

We then have by analogy with equation (99) that:

$$\Pi_V(q^2) = \frac{1}{g_5^2 q^2}\frac{\partial_z V(q,z)}{z}|_{z=\epsilon} \tag{109}$$

We will now use a different approach using Green's functions to solve for Π so that we may compare and glean information about the pion decay constant. We have the equation of motion for the vector field according to:

$$\partial_z\left(\frac{1}{z}\partial_z V(q,z)\right) + \frac{q^2}{z}V(q,z) = 0 \tag{110}$$

subject to the condition $V(q,\epsilon) = 1$ and $\partial_z V(q,z)|_{z_m} = 0$. We can substitute a Green's function $G(q,z,z')$ into this equation to find:

$$\partial_z(\frac{1}{z}\partial_z G(q,z,z')) + \frac{q^2}{z}G(q,z,z') = \delta(z-z') \tag{111}$$

The Green's function G is now subject to the conditions $G(q,\epsilon,z') = 0$ and $\partial_z G(q,z,z')|_{z_m} = 0$. We can now make one more substitution considering a $\psi(z)$:

$$\partial_z(\frac{1}{z}\partial_z \psi_n(z)) + \frac{m_n^2}{z}\psi_n(z) = 0 \tag{112}$$

$\psi(z)$ is then subject to the conditions $\psi_n(\epsilon) = \psi_n'(z_m) = 0$. This is a Sturm-Liouville problem, which we

show and confirm the orthogonality condition for by:

$$\sum_n [\int_\epsilon^{z_m} \frac{dz}{z} \psi_m(z)\psi_n(z)^* = \delta_{mn}]\psi_n(z') \tag{113}$$

$$\int_\epsilon^{z_m} \frac{dz}{z} \psi_m(z) \sum_n \psi_n(z)^*\psi_n(z') = \psi_m(z') \tag{114}$$

We then see from the orthogonality condition relation that

$$\sum_n \frac{\psi_n(z)^*\psi_n(z')}{z} = \delta(z - z') \tag{115}$$

To solve the equations, we make an ansatz according to:

$$G(q, z, z') = \sum_n \frac{\psi_n(z)\psi_n(z')^*}{q^2 - m_n^2} \tag{116}$$

We plug this into the equation for G above to find:

$$\partial_z(\frac{1}{z}\partial_z G(q, z, z')) + \frac{q^2}{z} G(q, z, z') = \sum_n \frac{\psi_n(z')^*}{q^2 - m_n^2} [\partial_z(\frac{1}{z}\partial_z \psi_n(z)) + \frac{q^2}{z}\psi_n(z)] \tag{117}$$

We then note that by the equation of motion for ψ, we can substitute to find:

$$\partial_z(\frac{1}{z}\partial_z G(q, z, z')) + \frac{q^2}{z} G(q, z, z') = \sum_n \frac{\psi_n(z')^*}{q^2 - m_n^2} [\frac{-m_n^2}{z}\psi_n(z) + \frac{q^2}{z}\psi_n(z)] \tag{118}$$

By the orthogonality condition found above, we then find:

$$\partial_z(\frac{1}{z}\partial_z G(q, z, z')) + \frac{q^2}{z} G(q, z, z') = \sum_n \frac{\psi_n(z)^*\psi_n(z')}{z} = \delta(z - z') \tag{119}$$

Thus, we have proved that our ansatz fulfills the equation of motion for G. We may next solve for $V(q, z)$ by considering the following integral:

$$\int_\epsilon^{z_m} dz V(q, z)[\partial_z(\frac{1}{z}\partial_z) + \frac{q^2}{z}]G(q, z, z') = V(q, z') = -\frac{\partial_z G(q, z, z')}{z}|_{z=\epsilon} \tag{120}$$

Where the first equality is true by the delta function nature of the Green's function equation of motion

and the second equality is from integrating by parts. We thus have:

$$V(q,z) = \frac{-1}{\epsilon}\sum_n \frac{\psi_n'(\epsilon)\psi_n(z)}{q^2 - m_n^2} \tag{121}$$

Now that we have an expression for V, we can take the derivative according to equation (109) (and switch back to A as promised) and find:

$$\Pi_A(-q^2) = \frac{-1}{g_5^2 q^2}\frac{\partial_z A(q,z)}{z}|_{z=\epsilon} = \frac{-1}{g_5^2 q^2}\sum_n \frac{(\frac{\psi_n'(\epsilon)}{\epsilon})^2}{(q^2 - m_n^2)m_n^2} \tag{122}$$

In the chiral limit, we have $m_\pi \to 0$, but $m_{a_1} \neq 0$. If we then consider $q^2 \to 0$, we have a pole in the Π function which allows us to pick out the pion decay constant. As $q^2 \to 0$, we find $\Pi_A(-q^2) \to \frac{f_\pi^2}{q^2}$. This limit allows us to find the final form of the pion decay constant which we will use in our derivation of the GOR relation:

$$f_\pi^2 = \frac{-1}{g_5^2}\frac{\partial_z A(0,z)}{z}|_{z=\epsilon} \tag{123}$$

7 Deriving the GOR Relation from the Equations of Motion

We must solve our system of three equations of motion, which are:

$$\partial_z\left(\frac{1}{z}\partial_z A^\mu(q,z)\right) + \frac{q^2}{z}A^\mu(q,z) - \frac{v(z)^2 g_5^2}{z^3}A^\mu(q,z) = 0 \tag{124}$$

$$\partial_z\left(\frac{1}{z}\partial^z\partial^\nu\phi^a(q,z)\right) + \frac{g_5^2 v(z)^2}{z^3}(\pi^a - \phi^a(q,z)) = 0 \tag{125}$$

$$\frac{g_5^2 v(z)^2}{z^2}\partial_z\pi^a - q^2\partial_z\phi^a = 0 \tag{126}$$

We construct a solution perturbatively, letting $q^2 = m_\pi^2$. From the similarity between equations (124) and (125), we try a solution of the form:

$$\phi^a = A^a(0,z) - 1 \tag{127}$$

With this ansatz, equation (126) tells us:

$$0 = -m_\pi^2 \partial_z (A^a(0,z) - 1) + \frac{g_5^2 v(z)^2}{z^2} \partial_z \pi^a \tag{128}$$

We can now solve directly for π^a:

$$\pi^a = m_\pi^2 \int_0^z \frac{u^2}{g_5^2 v(z)^2} \partial_z A^a(0,u) du = m_\pi^2 \int_0^z \frac{u^3}{v(z)^2} \frac{1}{g_5^2 u} \partial_z A^a(0,u) du \tag{129}$$

From our digression into defining the decay constant, we have

$$f_\pi^2 = -\frac{1}{g_5^2} \frac{\partial_z A(0,z)}{z}|_{z=\epsilon} \tag{130}$$

The function $\frac{u^3}{v^2}$ has a significant contribution only for small u, so we may replace the second part of the integrand in equation (129), which gives us:

$$\pi^a = -m_\pi^2 f_\pi^2 \int_0^z \frac{u^3}{v^2} du = -m_\pi^2 f_\pi^2 \int_0^z \frac{u^3 du}{m_q^2 u^2 + 2 m_q \sigma u^4 + \sigma^2 u^6} \tag{131}$$

Performing the integral then gives:

$$\pi^a = \frac{-m_\pi^2 f_\pi^2}{2 m_q \sigma} \tag{132}$$

We now find π^a by plugging our perturbative solution into equation (125) with $q^2 = 0$:

$$\partial_z \left(\frac{1}{z} \partial_z (A_\mu^a - 1) \right) + \frac{g_5^2 v^2}{z^3} (\pi^a - A_\mu^a + 1) = 0 \tag{133}$$

Solving equations (124) and (133) as a system with $q^2 = 0$ tells us:

$$-A = \pi - A + 1 \tag{134}$$

$$\pi = -1 \tag{135}$$

With the π field now known, we can plug back into equation (132):

$$2m_q\sigma = m_\pi^2 f_\pi^2 \tag{136}$$

This is precisely the desired form of the Gell-Mann-Oakes-Renner Relation.

8 Deriving the GOR for a Complex Chiral Condensate

The action for the theory in its full form was given by:

$$S = \int d^5x\sqrt{g}Tr\{|DX|^2 + 3\,|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2)\} \tag{137}$$

Definitions for the functions remain as previously. We again expand the action in terms of these functions to find:

$$\begin{aligned} S = \int d^5x\sqrt{g}Tr\{|\partial_M X|^2 - iL_M^b(\partial_M X^\dagger)T^b X + iR_M^b(\partial_M X^\dagger)XT^b + L_M^a L_M^b X^\dagger T^a T^b X + \\ iL_M^a X^\dagger T^a(\partial_M X) - L_M^a R_M^b X^\dagger T^a X T^b - iR_M^a T^a X^\dagger \\ (\partial_M X) - R_M^a L_M^b T^a X^\dagger T^b X + R_M^a R_M^b T^a X^\dagger X T^b + 3\,|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2)\} \end{aligned} \tag{138}$$

Let us first consider the term $|\partial_M X|^2 = (\partial_M X)(\partial^M X)$. The field X is given by $X = X_0 exp(2i\pi^c T^c)$. We no longer have $v(z) \in \Re$, so we cannot again use the unitary matrix trick from before to cancel terms. We instead expand immediately and cancel directly. We expand according to:

$$X = X_0(1 + 2i\pi^c T^c - 2\pi^c T^c \pi^b T^b) \tag{139}$$

The background field X_0 is now a complex function of our fifth-dimension parameter, z:

$$X_0 = \frac{v(z)}{2} = \frac{1}{2}m_q z + \frac{1}{2}\sigma z^3 \tag{140}$$

where now $\sigma \in C$. We expand, contracting indices and keeping only up to quadratic terms, to find:

$$
\begin{aligned}
Tr\{|\partial_M X|^2\} \approx \quad & z^2(Tr[|\partial_M X_0|^2 - 2i\pi^a T^a |\partial_M X_0|^2 - 2\pi^a T^a \pi^b T^b |\partial_M X_0|^2 \\
& -2i(\partial_M \pi^a) T^a (\partial_M X_0) X_0^\dagger - 2(\partial_M \pi^a) T^a \pi^b T^b (\partial_M X_0) X_0^\dagger - 2\pi^a T^a (\partial_M \pi^b) T^b (\partial_M X_0) X_0^\dagger \\
& +2i\pi^c T^c |\partial_M X_0|^2 + 4\pi^c T^c \pi^a T^a |\partial_M X_0|^2 + 4\pi^c T^c (\partial_M \pi^a) T^a X_0^\dagger (\partial_M X_0) \\
& -2\pi^c T^c \pi^d T^d |\partial_M X_0|^2 + 2i(\partial_M \pi^c) T^c X_0 (\partial_M X_0^\dagger) + 4(\partial_M \pi^c) T^c \pi^a T^a X_0 (\partial_M X_0^\dagger) \\
& +4(\partial_M \pi^c) T^c (\partial_M \pi^a) T^a |X_0|^2 - 2(\partial_M \pi^c) T^c \pi^d T^d X_0 (\partial_M X_0^\dagger) \\
& -2\pi^c T^c (\partial_M \pi^d) T^d X_0 (\partial_M X_0^\dagger)])
\end{aligned} \tag{141}
$$

Now, we note that $Tr\{T^a T^b\} = \frac{1}{2}\delta^{ab}$ and $Tr\{T^a\} = 0$, which allows us to simplify. After contracting indices in this manner, we consider all dummy indices among different terms equal and term them a so they can be combined:

$$
\begin{aligned}
Tr\{|\partial_M X|^2\} \approx \quad & |\partial_M X_0|^2 - \pi^a \pi^a |\partial_M X_0|^2 - (\partial_M \pi^a) \pi^a (\partial_M X_0) X_0^\dagger - \pi^a (\partial_M \pi^a) (\partial_M X_0) X_0^\dagger \\
& +2\pi^a \pi^a |\partial_M X_0|^2 + 2\pi^a (\partial_M \pi^a) (\partial_M X_0) X_0^\dagger - \pi^a \pi^a |\partial_M X_0|^2 + 2(\partial_M \pi^a) \pi^a X_0 (\partial_M X_0^\dagger) \\
& +2(\partial_M \pi^a)(\partial_M \pi^a) |X_0|^2 - (\partial_M \pi^a) \pi^a X_0 (\partial_M X_0^\dagger) - \pi^a (\partial_M \pi^a) X_0 (\partial_M X_0^\dagger)
\end{aligned} \tag{142}
$$

Combining terms, we have:

$$
Tr\{|\partial_M X|^2\} \approx |\partial_M X_0|^2 + 2(\partial_M \pi^a)(\partial_M \pi^a) |X_0|^2 \tag{143}
$$

The first term here is again background, so we neglect it to finally have:

$$
Tr\{|\partial_M X|^2\} \approx \frac{1}{2} m_q^2 z^4 (\partial_M \pi^a)(\partial_M \pi^a) + \frac{1}{2} m_q (\sigma + \sigma^*) z^6 (\partial_M \pi^a)(\partial_M \pi^a) + \frac{1}{2} \sigma \sigma^* z^8 (\partial_M \pi^a)(\partial_M \pi^a) \tag{144}
$$

We now examine the terms involving L and R. The approximations developed in section 4 were kept completely general (*i.e.* did not depend on the real or complex nature of σ) up to equations (65) and (69).

We can thus use these same equations and derive the new form of these terms, a replacement for equation (71):

$$\begin{aligned}
&\Gamma_{No\ \partial} + \Gamma_{\partial} = \\
&\frac{1}{2}m_q^2 z^4 A^{Ma}A^{Ma} + \frac{1}{2}m_q(\sigma+\sigma^*)z^6 A^{Ma}A^{Ma} + \frac{1}{2}\sigma\sigma^* z^8 \\
&-m_q^2 z^4(\partial_M\pi^a)A^{Ma} - m_q(\sigma+\sigma^*)z^6(\partial_M\pi^a)A^{Ma} - \sigma\sigma^* z^8(\partial_M\pi^a)A^{Ma}
\end{aligned} \tag{145}$$

Finally, we consider the last two terms. $|X|^2$ again only contributes to the background. Our evaluation of $F_L^2 + F_R^2$ was completely general and independent of the real or complex nature of σ, so we immediately have the previous result of:

$$Tr\{\frac{-1}{4g_5^2}(F_L^2 + F_R^2)\} \approx \frac{-z^4}{4g_5^2}F_A^a F_A^a \tag{146}$$

We can now write the final form of our approximated action by summing the results of the various simplifications, equations (144), (145), and (146). The approximated action for a complex σ is then:

$$\begin{aligned}
S = \quad & \int d^5x \left[\frac{1}{2z}m_q^2(\partial_M\pi^a)(\partial_M\pi^a) + \frac{1}{2}m_q(\sigma+\sigma^*)z(\partial_M\pi^a)(\partial_M\pi^a) + \frac{1}{2}\sigma\sigma^* z^3(\partial_M\pi^a)(\partial_M\pi^a)\right. \\
&+\frac{1}{2z}m_q^2 A^{Ma}A^{Ma} + (1/2)m_q(\sigma+\sigma^*)zA^{Ma}A^{Ma} + \frac{1}{2}\sigma\sigma^* z^3 A^{Ma}A^{Ma} - \frac{1}{z}m_q^2(\partial_M\pi^a)A^{Ma} \\
&\left.-m_q(\sigma+\sigma^*)z(\partial_M\pi^a)A^{Ma} - \sigma\sigma^* z^3(\partial_M\pi^a)A^{Ma} - \frac{1}{4g_5^2 z}F_A^a F_A^a\right]
\end{aligned} \tag{147}$$

Most of these terms can be factored together to give a simpler expression:

$$S = \int d^5x \left[-\frac{1}{4g_5^2 z}F_A^a F_A^a + \frac{|v(z)|^2}{2z^3}(\partial_M\pi^a - A_M^a)^2\right] \tag{148}$$

We can thus see immediately that the actions for σ either real or complex, equations (78) and (148), are equal if we let $\sigma, v(z) \in \Re$. We can then be confident in our result of the complex action and move on to deriving the GOR relation in the complex case. We never used the real or complex nature of the chiral

condensate until equation (131) above, so the entire derivation is the same up until that point. We then continue from there to find:

$$\pi^a = -m_\pi^2 f_\pi^2 \int_0^z \frac{u^3}{|v|^2} du = -m_\pi^2 f_\pi^2 \int_0^z \frac{u^3 du}{m_q^2 u^2 + m_q(\sigma + \sigma^*)u^4 + \sigma\sigma^* u^6} \tag{149}$$

The integral evaluates to:

$$\pi^a = -m_\pi^2 f_\pi^2 \left[\frac{ln(\sigma u^2 + m_q) - ln(\sigma^* u^2 + m^q)}{2m_q(\sigma - \sigma^*)} \right]_0^z \tag{150}$$

We may now rearrange and evaluate the limits. We take the z value to be much larger than our mass, effectively infinity. We then have:

$$\pi^a = -m_\pi^2 f_\pi^2 \left(\frac{ln(\frac{\sigma + \frac{m_q}{z^2}}{\sigma^* + \frac{m_q}{z^2}})}{2m_q(\sigma - \sigma^*)} \right) \rightarrow -m_\pi^2 f_\pi^2 \left(\frac{ln(\frac{\sigma}{\sigma^*})}{2m_q(\sigma - \sigma^*)} \right) \tag{151}$$

We again have $\pi^a = -1$, which gives us our version of the complex GOR relation:

$$m_\pi^2 f_\pi^2 ln(\frac{\sigma}{\sigma^*}) = 2m_q(\sigma - \sigma^*) \tag{152}$$

This is an interesting result, because it directly relates the phase of the chiral condensate to the imaginary part. Both sides of this equation are imaginary, so it is in fact valid. This complex GOR relation does in fact reduce to the original GOR relation in the limit $\sigma^* \rightarrow \sigma$:

$$\begin{aligned} \lim_{\sigma^* \rightarrow \sigma} \left(1 = \frac{2m_q(\sigma - \sigma^*)}{m_\pi^2 f_\pi^2 ln(\frac{\sigma}{\sigma^*})} \right) \\ 1 = \frac{-2m_q}{-m_\pi^2 f_\pi^2 \frac{1}{\sigma}} \\ 2m_q\sigma = m_\pi^2 f_\pi^2 \end{aligned} \tag{153}$$

There is a problem, though, in that our complex GOR relation does not match what has been found by the Chiral Lagrangian Method. The Chiral Lagrangian gives

$$m_\pi^2 f_\pi^2 = 2 m_q Re(\sigma) \tag{154}$$

Both methods are incomplete, since both take the phase between the mass of the pion and the chiral condensate to be a parameter, while in the Lagrangian of the Standard Model this phase can be solved for explicitly. Regardless, our method input the same chiral symmetry requirements as the Chiral Lagrangian, so we should find the same GOR Relation they do.

9 Higgs Enters the Picture

Thus far, we have ignored the Higgs field, since our aim is to only consider fluctuations of the pion field about the chiral condensate. We now postulate a solution as to why we produce an incorrect GOR relation. It seems possible that in our decomposition of the field X, we may have mixed the fluctuations of the pion field with those of the Higgs field. We multiplied the background $X_0 = \frac{1}{2}(m_q z + \sigma z^3)$, which contains both the quark mass and the chiral condensate, by the unitary matrix U, which contains the pion fluctuations about the chiral condensate. It is likely that we should only have multiplied the chiral condensate term by U to isolate the pion fluctuations as separate from the mass fluctuations, governed by the Higgs field. We will thus try to solve our problem by considering a new decomposition:

$$X = \frac{1}{2} m_q z + \frac{1}{2} \sigma z^3 U \tag{155}$$

It may seem strange that mixing the Higgs field in could allow us to still obtain the GOR relation in the real case, so we now check if this is possible using the four dimensional model our 5D model emulates through AdS/CFT. Let us consider the implications of including the Higgs field along with the pion fluctuations in a 4D Lagrangian. We will analyze a basic 4D Lagrangian which includes both kinetic terms for the Higgs and Quark fields as well as a term which couples the two, the Yukawa coupling term found above in the Introduction. We will force both fields to use the same fluctuation matrix U, which we believe was our mistake, and compare the resulting Lagrangian to what would occur if we had separate fluctuations, as happens in the intended Chiral Lagrangian case. The Lagrangian is:

$$\mathcal{L} = Tr[(\partial_\mu \Sigma^\dagger)(\partial^\mu \Sigma) + V(\Sigma^\dagger \Sigma) + (\partial_\mu H^\dagger)(\partial^\mu H) + V(H^\dagger H) + a(H\Sigma^\dagger + \Sigma H^\dagger)] \tag{156}$$

Here H is the Higgs field, Σ is the field containing the quarks, and a is the coefficient of the Higgs-pion coupling. Their vacuum expectation values and decompositions are as follows:

$$\langle \Sigma \rangle = \sigma \tag{157}$$

$$\Sigma = \sigma U^\dagger \tag{158}$$

$$\langle H \rangle = v \tag{159}$$

$$H = vU \tag{160}$$

We can now use these decompositions in terms of the unitary matrix U to find a Lagrangian that is solely in terms of U. We proceed to find it as follows:

$$\mathcal{L} = Tr[(|\sigma|^2 + v^2)(\partial_\mu U^\dagger)(\partial^\mu U) + av(\sigma^* U^2 + \sigma U^{\dagger 2})] + \text{constants} \tag{161}$$

We now decompose the Lagrangian in the same way, but allowing $H = v$, proportional to the identity matrix instead. This is the Chiral Lagrangian case, which yields:

$$\mathcal{L} = Tr[(|\sigma|^2)(\partial_\mu U^\dagger)(\partial^\mu U) + av(\sigma^* U + \sigma U^\dagger)] + \text{constants} \tag{162}$$

We may now compare equations (161) and (162). We expect that making the Higgs field independent of U removes the v dependence in the kinetic term. The different dependence in the U matrices in the potential would not change our GOR relation, though: since U commutes with itself, we can add exponents, giving $U^2 = exp(4i\pi^a T^a)$. The differing dependence would only add a factor of 2 to our GOR relation, since our original U was $exp(2i\pi^a T^a)$, not change its functional dependence as we suspected. We thus find that implicit mixing with the Higgs was not our problem. We proceed with the new decomposition mentioned anyway because it seems more valid for the same arguments mentioned above.

10 A New X Field Decomposition

Our new decomposition, as shown above, is:

$$X = \frac{1}{2} m_q z + \frac{1}{2} \sigma z^3 U \tag{163}$$

The action and its expansion remain as before in equation (53). We again consider each set of terms separately. First, using the same approximations as earlier (fields to quadratic order) we find for the first set of terms after cancellations:

$$Tr\{|\partial_M X|^2\} = \frac{3}{4} m_q(\sigma + \sigma^*) z^2 \pi^c \pi^c + \frac{1}{2} \sigma^* \sigma z^6 (\partial_M \pi^c)(\partial^M \pi^c) - \frac{1}{2} m_q(\sigma + \sigma^*) z^5 \pi^c (\partial_z \pi^c) \tag{164}$$

We can now consider the gauge field terms from equation (53). They still combine into the forms found earlier in equations (65) and (69). With our new decomposition, these expressions become:

$$2|X|^2 A^a_M A^{Ma} = \frac{1}{2} m_q^2 z^2 A^a_M A^{Ma} + \frac{1}{2} m_q z^4 (\sigma + \sigma^*) A^a_M A^{Ma} + \frac{1}{2} \sigma^* \sigma z^6 A^a_M A^{Ma} \tag{165}$$

$$2iTr\{A^b T^b((\partial_M X)X^\dagger - (\partial_M X^\dagger)X)\} = -m_q(\sigma+\sigma^*) z^3 \pi^c A^{Mc} - \frac{1}{2} m_q(\sigma+\sigma^*) z^4 (\partial_M \pi^c) A^{Mc} - \sigma^* \sigma z^6 (\partial_M \pi^c) A^{Mc} \tag{166}$$

Previously, with our old decomposition, the $|X|^2$ term was simply background because the unitary matrix canceled with itself and left the whole term proportional to the identity matrix. This cancellation does not occur with the new decomposition, so we must evaluate it, which gives:

$$\sqrt{g} Tr\{3|X|^2\} = -\frac{3}{4z} m_q(\sigma + \sigma^*) \pi^c \pi^c \tag{167}$$

We may now combine all these terms into our full action:

$$S = \int d^5x(-\frac{1}{2}m_q(\sigma+\sigma^*)\pi^c(\partial_z\pi^c) + \frac{1}{2}\sigma^*\sigma z(\partial_M\pi^c)(\partial^M\pi^c) + \frac{1}{2z^3}m_q^2A_M^cA^{Mc} + \frac{1}{2z}m_q(\sigma+\sigma^*)A_M^cA^{Mc} + \frac{1}{2}\sigma^*\sigma z A_M^cA_M^c - \frac{1}{2z}m_q(\sigma+\sigma^*)(\partial_M\pi^c)A^{Mc} - \sigma^*\sigma z(\partial_M\pi^c)A^{Mc} - \frac{1}{4g_5^2z}F_A^cF_A^c) \quad (168)$$

This is our action for the new decomposition $X = \frac{1}{2}m_qz + \frac{1}{2}\sigma z^3U$ in terms of a complex chiral condensate. We will derive the equations of motion for a real chiral condensate first, in the same manner we used for the old decomposition. We will take the same $A_z = 0$ gauge and use the same e^{-iqx} ansatz. The Euler-Lagrange equations for the transverse part of the gauge field, A, and longitudinal part of the gauge field ϕ, and the pion field, π, are as follows:

$$\partial_z\left(\frac{1}{z}\partial_zA^\mu\right) + \frac{q^2}{z}A^\mu - \frac{g_5^2m_q^2}{z^3}A^\mu - 2m_q\sigma z g_5^2A^\mu - g_5^2\sigma^2zA^\mu = 0 \quad (169)$$

$$\partial_z\left(\frac{1}{z}\partial_z\phi^c\right) + \frac{g_5^2m_q^2}{z^3}\phi^c + 2m_q\sigma z g_5^2\phi^c + g_5^2\sigma^2z(\phi^c-\pi^c) - \frac{g_5^2m_q\sigma}{z}\pi^c = 0 \quad (170)$$

$$\left(\frac{q^2}{g_5^2}+m_q\sigma\right)\partial_z\left(\frac{1}{z}\partial_z\phi^c\right) + \frac{q^2m_q^2}{z^3}\phi^c + 2q^2m_q\sigma z\phi^c + \sigma^2\partial_z(z\partial_z(\phi^c-\pi^c)) - \frac{m_q\sigma g_5^2}{z}\pi^c - \frac{q^2m_q\sigma}{z}\phi^c = 0 \quad (171)$$

These are our equations of motion. They remain elusive to analytic solution. Despite this, we have successfully developed a conjecture for a possible solution to troubling issues concerning pion condensation and the form of the complex GOR relation. We hope that these solutions will bear fruit and help to establish a possible fix to the model to improve its accuracy in these areas.

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